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DIFFERENT TYPES OF IDEALS AND HOMOMORPHISMS OF (m, n)-SEMIRINGS

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ABSTRACT. In this article, we develop some more of the theory of (m, n)-semirings. In particular, we study ideals, primary ideals, and subtractive ideals of (m, n)-semirings and Γ-(m, n)-semirings. We describe the functions between (m, n)-semirings that preserve the (m, n)-semiring structure. Also, we look at another way of forming new (m, n)-semiring from existing ones.

Keywords: (m, n)-semiring, primary ideal, subtractive ideal, homomorphism.

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1. Introduction to (m, n)-semirings

The notion of a semiring was introduced by Vandiver in 1934 [19]. Semirings are studied by many authors in various directions. One of the main directions of such studies is investigation of properties of ideals, for example see [3, 4, 5, 8, 10, 18]. Crombez [6] in 1972 generalized rings and named it as (n, m)-rings. It was further studied by Crombez and Timm [7], Leeson and Butson [11, 12], Dudek [9], Mirvakili and Davvaz [13, 14, 15]. Alam, Rao and B. Davvaz [1] proposed a new class of mathematical structures called (m, n)-semirings (which generalize the usual semirings) and described their basic properties. They gave the definition of partial ordering and initiated the generalization of congruence and homomorphism for (m, n)-semirings. Also, see , Pop [16], Pop and Lauran [17], Asadi et al. [2].

Let R be a non-empty set and $f : R^m \to R$ be a map, that is, f is an m-ary operation. A non-empty set R with an m-ary operation f is called an m-ary groupoid and is denoted by (R, f). We use the following general convention. The sequence $x_i, x_{i+1}, \ldots, x_m$ is denoted by x_i^m where $1 \leq i \leq j \leq m$. For all $1 \leq i \leq j \leq m$, the following term $f(x_1, x_2, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_m)$ is represented as $f(x_1^i, y_{i+1}^j, z_{j+1}^m)$. In the case when $y_{i+1} = y_{i+2} = \ldots = y_j = y$, the term is expressed as $f(x_1^i, y^{(j-i)}, z_{j+1}^m)$. An m-ary groupoid (R, f) is called an m-ary semigroup if f is associative, that is, if $f(x_1^{i-1}, f(x_i^{m+i-1}), x_{m+i}^{2m-1}) = f(x_1^{j-1}, f(x_j^{m+j-1}), x_{m+j}^{2m-1})$, for all $x_1, x_2, \ldots, x_{2m-1} \in R$ where $1 \leq i \leq j \leq m$. We say f is commutative if

$$f(x_1, x_2, \ldots, x_m) = f(x_{\eta(1)}, x_{\eta(2)}, \ldots, x_{\eta(m)})$$

for every permutation η of $\{1, 2, \ldots, m\}$, $x_1, x_2, \ldots, x_m \in R$. Let R be a non-empty set and f, g be m-ary and n-ary operations on R, respectively. The n-ary operation g is distributive with respect to the m-ary operation f if

$$g(x_1^{i-1}, f(a_1^m), x_{i+1}^n) = f(g(x_1^{i-1}, a_1, x_{i+1}^n), \ldots, g(x_1^{i-1}, a_m, x_{i+1}^n)),$$

for every a_1^m, x_1^n in R and $1 \leq i \leq n$. An m-ary semigroup (R, f) is called a semi-abelian or $(1, m)$-commutative if

$$f(x, \underbrace{a, \ldots, a}_{m-2}, y) = f(y, \underbrace{a, \ldots, a}_{m-2}, x).$$

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for all $a, x, y \in R$.

Definition 1.1. Let R be a non-empty set and f, g be m-ary and n-ary operations on R, respectively. Then (R, f, g) is called an (m, n)-*semiring* if the following conditions hold:

(1) (R, f) is an m-ary semigroup;
(2) (R, g) is an n-ary semigroup;
(3) The n-ary operation g is distributive with respect to the m-ary operation f.

One can find many examples of (m, n)-semirings in [1].

Let (R, f, g) be an (m, n)-semiring. Then, m-ary semigroup (R, f) has an identity element 0 if

$$x = f(\underbrace{0, \ldots, 0}_{i-1}, x, \underbrace{0, \ldots, 0}_{m-i}),$$

for all $x \in R$ and $1 \leq i \leq m$. We call 0 as an identity element of (m, n)-semiring (R, f, g). Similarly, n-ary semigroup (R, g) has an identity element 1 if

$$y = g(\underbrace{1, \ldots, 1}_{j-1}, y, \underbrace{1, \ldots, 1}_{n-j}),$$

for all $y \in R$ and $1 \leq j \leq n$.

2. Ideals of (m, n)-semirings

In this paper f is an addition m-ary operation and g is a multiplication n-ary operation.

Definition 2.1. Let I be a non-empty subset of an (m, n)-semiring (R, f, g) and $1 \leq i \leq n$. We call I an i-ideal of R if

(1) I is a subsemigroup of m-ary semigroup (R, f);
(2) For every $a_1, a_2, \ldots, a_n \in R$, $g(a_1, a_2, \ldots, a_{i-1}, I, a_{i+1}, \ldots, a_n) \subseteq I$.

I is called an ideal of R if for every $1 \leq i \leq n$, I is an i-ideal.

Lemma 2.1. If $A_1, \ldots, A_n$ are ideals of (m, n)-semiring (S, f, g), then

(1) $A_1 \cap \ldots \cap A_n$ is an ideal of (S, f, g);
(2) $f(A_1, \ldots, A_m)$ is an ideal of (S, f, g);
(3) $g(A_1, \ldots, A_n)$ is an ideal of (S, f, g).

Definition 2.2.

(1) A proper ideal I of an (m, n)-semiring (R, f, g) is said to be *prime* if for any ideals $A_1, \ldots, A_n$ of R, $g(A_1, \ldots, A_n) \subseteq I$ implies $A_i \subseteq I$ for some $1 \leq i \leq n$.
(2) A proper ideal I of an (m, n)-semiring (R, f, g) is said to be *weakly prime* if for any ideals $A_1, \ldots, A_n$ of R, $\{0\} \neq g(A_1, \ldots, A_n) \subseteq I$ implies $A_i \subseteq I$ for some $1 \leq i \leq n$.
(3) An ideal I of an (m, n)-semiring (R, f, g) is called *subtractive* or k-*ideal* if for any elements $a_1, \ldots, a_{n-1} \in I$ and $a_n \in R$, $g(a_1, \ldots, a_n) \in I$, then $a_n \in I$.

Theorem 2.1. *An ideal of an (m, n)-semiring (S, f, g) is weakly prime if and only if for any ideals $A_1, A_2, \ldots, A_n$ of S, we have:*

either $g(A_1, A_2, \ldots, A_n) = A_1$ *or* $\ldots$ *or* $g(A_1, A_2, \ldots, A_n) = A_n$ *or* $g(A_1, A_2, \ldots, A_n) = 0$.

Proof. Suppose that every ideal of S is weakly prime. Let $A_1, A_2, \ldots, A_n$ be ideals of S. If $g(A_1, A_2, \ldots, A_n) \neq S$, then $g(A_1, A_2, \ldots, A_n)$ is weakly prime. If $\{0\} \neq g(A_1, A_2, \ldots, A_n) \subseteq g(A_1, A_2, \ldots, A_n)$, then we have $A_i \subseteq g(A_1, A_2, \ldots, A_n)$ for some i (since $g(A_1, A_2, \ldots, A_n)$ is weakly prime ideal of S). Hence, $A_i = g(A_1, A_2, \ldots, A_n)$ for some i. If $g(A_1, A_2, \ldots, A_n) = S$, then $A_1 = A_2 = \ldots = A_n = S$.

Conversely, let I be any proper ideal of S and suppose that $\{0\} \neq g(A_1, A_2, \ldots, A_n) \subseteq I$ for ideals $A_1, A_2, \ldots, A_n$ of S. Then, we have $A_i = g(A_1, A_2, \ldots, A_n) \subseteq I$ for some i. $\square$

Lemma 2.2. Let P be a subtractive ideal of (m,n)-semiring (S,f,g). Let P be a weakly prime ideal but not a prime ideal of S. If $g(a_1, a_2, \ldots, a_n) = 0$ for some $a_1, a_2, \ldots, a_n \notin P$, then

$$g(a_1, P^{(n-1)}) = g(P, a_2, P^{(n-2)}) = \ldots = g(P^{(n-1)}, a_n) = \{0\}.$$

Proof. Suppose that $g(a_1, p^{(n-1)}) \neq 0$ for some $p_1, p_2, \ldots, p_{n-1} \in P$. Then, we obtain

$$0 \neq g(a_1, f(g(1, a_2, a_3, \ldots, a_n), (g(1, p_1, p_2, \ldots, p_{n-1}))^{(m-1)}), 1^{(n-2)}) \in P.$$

Since P is a weakly prime ideal of S, it follows that $a_1 \in P$ or

$$f(g(1, a_2, a_3, \ldots, a_n), (g(1, p_1, p_2, \ldots, p_{n-1}))^{(m-1)}) \in P,$$

that is, $a_i \in P$ for some $1 \leq i \leq n$, a contradiction. Therefore, $g(a_1, P^{(n-1)}) = \{0\}$. Similarly, we can show that $g(P, a_2, P^{(n-2)}) = \ldots = g(P^{(n-1)}, a_n) = \{0\}$. $\square$

Theorem 2.2. *Let P be a subtractive ideal of an (m,n)-semiring (S,f,g). If P is a weakly prime ideal but not prime, then $P^n = \{0\}$.*

Proof. Suppose that $g(p_1, p_2, \ldots, p_n) \neq 0$ for some $p_1, p_2, \ldots, p_n \in P$ and $g(a_1, a_2, \ldots, a_n) = 0$ for some $a_1, a_2, \ldots, a_n \notin P$, where P is not a prime ideal of S. Then, by Lemma 2.5,

$$0 \neq g(f(a_1, p_1^{(m-1)}), f(p_2, a_2, p_2^{(m-2)}), \ldots, f(a_n, p_n^{(m-1)})) \in P.$$

Hence, either $f(a_1, p_1^{(m-1)}) \in P$ or $f(p_2, a_2, p_2^{(m-2)}) \in P$ or ... or $f(a_n, p_n^{(m-1)}) \in P$, and so $a_i \in P$ for some $1 \leq i \leq n$, a contradiction. Hence, $P^n = \{0\}$. $\square$

Corollary 2.1. *Let P be a weakly prime ideal of (m,n)-semiring (S,f,g). If P is not a prime ideal of S, then $P \subseteq Nil\ S$.*

A subtractive ideal in a commutative (m,n)-semiring (S,f,g), satisfying $P^n = \{0\}$ may not be weakly prime.

Lemma 2.3. Let h be a homomorphism from (m,n)-semiring (S_1, f, g) onto (m,n)-semiring (S_2, f', g'). Then, each of the following statements is true:

(1) If I is an ideal (subtractive ideal) in S_1, then $h(I)$ is an ideal (subtractive ideal) in S_2.
(2) If J is an ideal (subtractive ideal) in S_2, then $h^{-1}(J)$ is an ideal (subtractive ideal) in S_1.

Theorem 2.3. *If $h : S_1 \longrightarrow S_2$ is a homomorphism of (m,n)-semirings and P is a prime ideal of S_2, then $h^{-1}(P)$ is a prime ideal of S_1.*

Proof. By the previous lemma $h^{-1}(P)$ is an ideal of (S_1, f, g). Let $g(a_1, a_2, \ldots, a_n) \in h^{-1}(P)$. Then, $h(g(a_1, a_2, \ldots, a_n)) \in P$ implies $g'(h(a_1), h(a_2), \ldots, h(a_n)) \in P$. Since P is a prime ideal of S_2, it follows that $h(a_i) \in P$ for some $1 \leq i \leq n$. Thus, $a_i \in h^{-1}(P)$ for some $1 \leq i \leq n$. Hence, $h^{-1}(P)$ is a prime ideal of S_1. $\square$

Theorem 2.4. *Let (S,f,g) be an (m,n)-semiring such that $S = \langle a_1, a_2, \ldots, a_k \rangle$ for $k = max\{n,m\}$ is a finitely generated ideal of S. Then, each proper k-ideal A of S is contained in a maximal k-ideal of S.*

Proof. Let β be the set of all k-ideals B of S satisfying $A \subseteq B \subset S$, partially ordered by inclusion. Consider a chain $\{B_i \mid i \in I\}$ in β. One easily checks that $B = \bigcup B_i$ is a k-ideal of S, because if $a_1, a_2, \ldots, a_{n-1}, f(a_1, a_2, \ldots, a_n) \in B$ then as defined B, there is $i_1, i_2, \ldots, i_{n-1}, j \in I$ such that $a_1 \in B_{i_1}, a_2 \in B_{i_2}, \ldots, a_{n-1} \in B_{i_{n-1}}, f(a_1, a_2, \ldots, a_n) \in B_j$, as B_i partially ordered by inclusion, then $B_j \subseteq B_{i_1}$ or $B_{i_1} \subseteq B_j$. Without loss of generality assuming that $B_{i_1}, B_{i_2}, \ldots, B_{i_{n-1}} \subseteq B_j$, then $a_1, a_2, \ldots, a_{n-1}, f(a_1, a_2, \ldots, a_n) \in B_j$ because B_j is a k-ideal. Therefore, $a_n \in B_j$ and $B_j \subseteq B$; so $a_n \in B$ which means B is a k-ideal, and $S = \langle a_1, a_2, \ldots, a_k \rangle$ implies $B \neq S$, and hence $B \in \beta$. By Zorn's lemma, β has a maximal element as we were to show. $\square$

Corollary 2.2. *Let (S, f, g) be an (m, n)-semiring with identity 1. Then, each proper k-ideal of S is contained in a maximal k-ideal of S.*

Proof. The proof is immediate by $S = \langle 1 \rangle$. □

Lemma 2.4. If A, B are two k-ideals of an (m, n)-semiring (S, f, g), then $A \cap B$ is a k-ideal.

Proof. Suppose that A, B are two k-ideals of S. Then, $A \cap B$ is an ideal. Now, let $x \in S$ such that $f(a_1^{m-1}, x) \in A \cap B$ for some $a_1, a_2, \ldots, a_{m-1} \in A \cap B$. Then $a_1, a_2, \ldots, a_{m-1} \in A$, $a_1, a_2, \ldots, a_{m-1} \in B$, $f(a_1^{m-1}, x) \in B$ and $f(a_1^{m-1}, x) \in A$. So, $x \in A$ and $x \in B$ as A, B are k-ideals. Hence, $x \in A \cap B$. □

Definition 2.3. An equivalence relation ρ on an (m, n)-semiring (S, f, g) is called a *congruence* on S if for any $a_1, \ldots, a_m, b_1, \ldots, b_n \in S$ such that $a\rho b$, then

(1) $f(a, a_2^m)\rho f(b, a_2^m)$;
(2) $g(a, b_2^n)\rho g(b, b_2^n)$;
(3) $g(b_2^n, a)\rho g(b_2^n, b)$.

Let ρ be a congruence on an (m, n)-semiring (S, f, g). Then, the congruence class of $x \in S$ is denoted by $x\rho$ and is defined by $x\rho = \{y \in S \mid (x, y) \in \rho\}$. The set of all congruence classes of S is denoted by S/ρ. Now, we define two operations on S/ρ as follows:

$$f(a_1\rho, \ldots, a_m\rho) = f(a_1^m)\rho \text{ and } g(b_1\rho, \ldots, b_n\rho) = g(b_1^n)\rho,$$

for all $a_1, \ldots, a_m, b_1, \ldots, b_n \in S$.

Theorem 2.5. *Let (S, f, g) be an (m, n)-semiring. Then, $(S/\rho, f, g)$ is an (m, n)-semiring under the above operations.*

Proof. Suppose that $a_1\rho, a_2\rho, \ldots, a_m\rho$ are elements of S/ρ. Then, for every permutation η at $\{1, 2, \ldots, m\}$,

$$\begin{aligned} f(a_1\rho, a_2\rho, \ldots, a_m\rho) &= f(a_1, \ldots, a_m)\rho = f(a_{\eta(1)}, a_{\eta(2)}, \ldots, a_{\eta(m)})\rho \\ &= f(a_{\eta(1)}\rho, a_{\eta(2)}\rho, \ldots, a_{\eta(m)}\rho). \end{aligned}$$

So, S/ρ is commutative under addition.

For each $1 \leq i \leq j \leq m$, we have

$$\begin{aligned} &f(a_1\rho, a_2\rho, \ldots, a_{i-1}\rho, f(a_i\rho, a_{i+1}\rho, \ldots, a_{m+i-1}\rho), a_{m+i}\rho, a_{m+i+1}\rho, a_{2m-1}\rho) \\ &= f(a_1\rho, a_2\rho, \ldots, a_{j-1}\rho, f(a_j\rho, a_{j+1}\rho, \ldots, a_{m+j-1}\rho), a_{m+j}\rho, a_{m+j+1}\rho, \ldots, a_{2m-1}\rho). \end{aligned}$$

So, addition is associative on S/ρ. Similarly, multiplication is associative.

Finally, we have the distributive law,

$$\begin{aligned} &g(a_1\rho, a_2\rho, \ldots, a_{i-1}\rho, f(b_1\rho, b_2\rho, \ldots, b_m\rho), a_{i+1}\rho, a_{i+2}\rho, \ldots, a_n\rho) \\ &= f(g(a_1\rho, a_2\rho, \ldots, a_{i-1}\rho, b_1\rho, a_{i+1}\rho, \ldots, a_n\rho), g(a_1\rho, a_2\rho, \ldots, a_{i-1}\rho, b_2\rho, a_{i+1}\rho, \ldots, a_n\rho), \\ &\ldots, g(a_1\rho, a_2\rho, \ldots, a_{i-1}\rho, b_m\rho, a_{i+1}\rho, \ldots, a_n\rho)). \end{aligned}$$

Therefore, S/ρ is an (m, n)-semiring. □

Lemma 2.5. Let (R, f, g) be an (m, n)-semiring with $1 \neq 0$. Then, R has at least one k-maximal ideal.

Proof. Since $\{0\}$ is a proper k-ideal of R, it follows that the set Δ of all proper k-ideals of R is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on Δ, and by using Zorn's lemma, a maximal k-ideal of R is just a maximal member of the partially ordered set $(\Delta, \subseteq)$. □

3. Primary ideal

Definition 3.1. Let (R, f, g) be an (m, n)-semiring and I be an ideal of R. The union of all ideals B such that $B^s \subseteq I$ for some positive integer l where $s = l(2n-1)$ or $s = l(2n+1)$ is an ideal of R and is called the *radical* of I which we shall denote by $N(I)$.

Definition 3.2. Let (R, f, g) be an (m, n)-semiring and I an ideal of R. The set of all elements $x \in R$ such that $x^s \in I$ for some positive integer l where $s = l(2n-1)$ or $s = l(2n+1)$ is said to be the *nil-radical* of I which we shall denote by $P(I)$.

If I is 0 in the previous definitions we use the symbols N and P for the radicals (radical and nil-radical) of 0.

From the above preliminary discussion and definitions, we introduce the following definition.

Definition 3.3. A proper ideal I of an (m, n)-semiring (R, f, g) is called *i-N-primary* provided $a_1, a_2, \ldots, a_n \in R$ with $g(a_1 \ldots a_n) \in I$ implies $a_i \in I$ or $j \neq i$ and $j \in \{1, 2, \ldots, n\}$, $a_j \in N(I)$.

The ideal I is said to be *N-primary* provided it is i-N-primary for all $i \in \{1, 2, \ldots, n\}$.

If we substitute the symbol P for N in the definition, we have the definitions of i-P-primary and P-primary.

Remark 3.1. It is clear that prime ideal in an (m, n)-semiring (R, f, g) is N-primary, but the converse is not true in general (similarly, for P-primary).

Definition 3.4. A proper ideal I of an (m, n)-semiring (R, f, g) is called *weakly i-N-primary* provided $a_1, a_2, \ldots, a_n \in R$ with $0 \neq g(a_1, a_2, \ldots, a_n) \in I$ implies $a_i \in I$ or $j \neq i$ and $j \in \{1, 2, \ldots, n\}$, $a_j \in N(I)$.

The ideal I is called *weakly N-primary* provided it is weakly i-N-primary for all $i \in \{1, 2, \ldots, n\}$.

If we substitute the symbol P for N in the definition, we have the definitions of weakly i-P-primary and weakly P-primary.

Remark 3.2. It is easy to see N-primary ideal is weakly N-primary, but the converse is not true, because 0 is always weakly N-primary ideal (by definition) but not necessarily N-primary. So, weakly N-primary ideal need not to be N-primary (similarly, for P-primary ideal).

Remark 3.3. It is clear that every weakly prime ideal of an (m, n)-semiring (R, f, g) is weakly N-primary, but the converse is not true in general (similarly, for weakly P-primary ideal).

Lemma 3.1. Let I be a weakly P-primary subtractive ideal of an (m, n)-semiring (R, f, g). If I is not a P-primary ideal, then $I^n = \{g(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in I\} = 0$.

Proof. Suppose that $I^n \neq 0$. We show that I is a P-primary ideal of R. Suppose that $g(a_1, a_2, \ldots, a_n) \in I$ where $a_1, a_2, \ldots, a_n \in R$. If $g(a_1, a_2, \ldots, a_n) \neq 0$, then there exist $i \in \{1, 2, \ldots, n\}$, $a_i \in I$ or $a_i \in P(I)$. Assume that $g(a_1, a_2, \ldots, a_n) = 0$. If $0 \neq g(a_1, a_2, \ldots, a_{n-1}, I) \subseteq I$, then there is an element d_n of I such that $g(a_1, a_2, \ldots, a_{n-1}, d_n) \neq 0$. Hence,

$$0 \neq g(a_1, a_2, \ldots, a_{n-1}, d_n) = g(a_1, a_2, \ldots, a_{n-1}, f(d_n, a_n^{(m-1)}) \in I.$$

Then, either $a_i \in I$ for $i \in \{1, 2, \ldots, n-1\}$ or $f(d_n, a_n^{(m-1)}) \in P(I)$. Thus, $a_i \in I$ for $i \in \{1, 2, \ldots, n-1\}$ or $a_n \in P(I)$. Therefore, I is a P-primary ideal.

Suppose that $g(a_1, a_2, \ldots, a_{n-1}, I) = 0$. If $g(a_1, a_2, \ldots, a_{n-2}, I, a_n) \neq 0$, then there exists $d_{n-1} \in I$ such that $g(a_1, a_2, \ldots, a_{n-2}, d_{n-1}, a_n) \neq 0$. Now, we have

$$0 \neq g(a_1, a_2, \ldots, a_{n-2}, f(a_{n-1}^{(m-1)}, d_{n-1}), a_n) \in I.$$

So, we obtain $a_i \in I$ for $i \in \{1, 2, \ldots, n-2, n\}$ or $a_{n-1} \in P(I)$, and hence I is a P-primary ideal. Thus, we assume that

$$g(a_1, a_2, \ldots, a_{n-2}, I, a_n) = 0.$$

Also, we can prove that $g(I, a_2, \ldots, a_{n-2}, a_{n-1}, a_n) = 0$. Since $I^n \neq 0$, it follows that there are elements $c_1, c_2, \ldots, c_n \in I$ such that $g(c_1, c_2, \ldots, c_n) \neq 0$. Then, $0 \neq g(c_1, c_2, \ldots, c_n) = g(f(a_1^{(m-1)}, c_1), f(a_2^{(m-1)}, c_2), \ldots, f(a_n^{(m-1)}, c_n) \in I$, so either $a_i \in I$ or $a_i \in P(I)$ for $i \in \{1, 2, \ldots, n\}$, and hence I is a P-primary ideal. $\square$

Theorem 3.1. *Let I be a proper subtractive ideal of an (m,n)-semiring (R,f,g). If for ideals $A_1, A_2, \ldots, A_n$ of R with $0 \neq g(A_1, A_2, \ldots, A_n) \subseteq I$ implies $A_i \subseteq I$ or for some positive integer k, $s = k(2n-1)$ or $s = k(2n+1)$, $A_i^s = \{a_i^s \in R | \ a_i \in A_i\} \subseteq I$, then I is a weakly P-primary ideal of R.*

Proof. Suppose that I is a proper subtractive ideal of an (m,n)-semiring (R,f,g) and let $0 \neq g(a_1, a_2, \ldots a_n) \in I$, where $a_1, a_2, \ldots, a_n \in R$. Then, $0 \neq g(\langle a_1\rangle, \langle a_2\rangle, \ldots, \langle a_n\rangle) \subseteq I$. Hence, $\langle a_i\rangle \subseteq I$ or $\langle a_i^s\rangle \subseteq I$ for some positive integer k, where $s = k(2n-1)$ or $s = k(2n+1)$. So, $a_i \in I$ or $a_i^s \in I$ for some positive integer k, where $s = k(2n-1)$ or $s = k(2n+1)$. This implies that $a_i \in P(I)$. Therefore, I is a weakly P-primary ideal of R. □

Lemma 3.2. If I is a weakly P-primary subtractive ideal that is not a P-primary over a semiring R, then $P(I) = P$.

Proof. Assume that I is a weakly P-primary subtractive ideal that is not a P-primary over an (m,n)-semiring (R,f,g). Then, it is clear that $P \subseteq P(I)$. Now, by Lemma 3.5, $I^n = 0$ gives $I \subseteq P$, and hence $P(I) \subseteq P$. Therefore, $P(I) = P$. □

4. Homomorphism of (m,n)-semirings

We recall the following definition from [1].

Definition 4.1. A mapping η from an (m,n)-semiring (R,f,g) into an (m,n)-semiring (R',f',g') is called a *homomorphism* if

$$\begin{aligned} g(a_1, a_2, \ldots, a_n)\eta &= g'(a_1\eta, a_2\eta, \ldots, a_n\eta), \\ f(a_1, a_2, \ldots, a_m)\eta &= f'(a_1\eta, a_2\eta, \ldots, a_m\eta), \end{aligned}$$

for each $a_1, \ldots, a_m \in R$.

An isomorphism is a one-to-one homomorphism. The semirings R and R' are called *isomorphic* (denoted by $R \cong R'$) if there exists an isomorphism from R onto R'.

Definition 4.2. A homomorphism η from the semiring (R,f,g) onto the semiring (R',f',g') is said to be *maximal* if for each $a \in R'$ there exists $c_a \in \eta^{-1}(\{a\})$ such that

$$f(x, ker(\eta)^{(m-1)}) \subset f(c_a, ker(\eta)^{(m-1)}),$$

for each $x \in \eta^{-1}(\{a\})$, where $ker(\eta) = \{x \in R \mid x\eta = 0\}$.

Lemma 4.1. Let η be a homomorphism from the semiring (R,f,g) onto the semiring (R',f',g'). If η is maximal, then $ker(\eta)$ is a Q-ideal, where $Q = \{c_a\}_{a \in R'}$.

Proof. It is clear that $\bigcup_{a\in R} f(c_a, ker(\eta)^{(m-1)}) = R$. Let c_a and c_b be distinct elements in Q and $a \neq b$. Assume that

$$f(c_a, ker(\eta)^{(m-1)}) \cap f(c_b, ker(\eta)^{(m-1)}) \neq \emptyset.$$

Thus, there exist $k_1, \ldots, k_{m-1}, k_1', \ldots, k'_{m-1} \in ker(\eta)$ such that $f(c_a, k_1^{m-1}) = f(c_b, k_1'^{m-1})$. Hence, we have

$$\begin{aligned} a &= f'(c_a\eta, k_1\eta, \ldots, k_{m-1}\eta) = (f(c_a, k_1, \ldots, k_{m-1}))\eta \\ &= (f(c_b, k_1', \ldots, k_{m-1}'))\eta = f'(c_b\eta, k_1'\eta, \ldots, k_{m-1}'\eta) = b, \end{aligned}$$

a contradiction. Now, it follows that $ker(\eta)$ is a Q-ideal. □

Lemma 4.2. Let R, R', η and Q be as stated in Lemma 4.3 and $c_{a_1}, c_{a_2}, \ldots, c_{a_m}, c_{a_{m+1}}$ elements in Q.

(1) If $f(f(c_{a_1}, \ldots, c_{a_m}), ker(\eta)^{(m-1)}) \subset f(c_{a_{m+1}}, ker(\eta)^{(m-1)})$, then $f(a_1, a_2, \ldots, a_m) = a_{m+1}$.

(2) If $f(g(c_{a_1}, c_{a_2}, \ldots, c_{a_n}), ker(\eta)^{(m-1)}) \subset f(c_{a_{n+1}}, ker(\eta)^{(m-1)})$, then $g(a_1, a_2, \ldots, a_n) = a_{n+1}$.

Proof. (1) Since

$$f(c_{a_1}, c_{a_2}, \ldots, c_{a_m}) \in f(f(c_{a_1}, c_{a_2}, \ldots, c_{a_m}), ker(\eta)^{(m-1)}) \subset f(c_{a_{m+1}}, ker(\eta)^{(m-1)}),$$

it follows that there exists $k_1, \ldots, k_{m-1} \in ker(\eta)$ such that $f(c_{a_1}, c_{a_2}, \ldots, c_{a_m}) = f(c_{a_{m+1}}, k_1^{m-1})$. Thus, we obtain

$$\begin{aligned} f'(a_1, a_2, \ldots, a_m) &= f'(c_{a_1}\eta, c_{a_2}\eta, \ldots, c_{a_m}\eta) = (f(c_{a_1}, c_{a_2}, \ldots, c_{a_m}))\eta \\ &= (f(c_{a_{m+1}}, k_1^{m-1}))\eta = f'(c_{a_{m+1}}\eta, k_1\eta, \ldots, k_{m-1}\eta) = a_{m+1}. \end{aligned}$$

(2) Since

$$g(c_{a_1}, c_{a_2}, \ldots, c_{a_n}) \in f(g(c_{a_1}, c_{a_2}, \ldots, c_{a_n}), ker(\eta)^{(m-1)}) \subseteq f(c_{a_{n+1}}, ker(\eta)^{(m-1)}),$$

it follows that there exists $k_1, \ldots, k_{m-1} \in ker(\eta)$ such that $g(c_{a_1}, c_{a_2}, \ldots, c_{a_n}) = f(c_{a_{n+1}}, k_1^{m-1})$. Thus, we have

$$\begin{aligned} g'(a_1, a_2, \ldots, a_n) &= g'(c_{a_1}\eta, c_{a_2}\eta, \ldots, c_{a_n}\eta) = (g(c_{a_1}, c_{a_2}, \ldots, c_{a_n}))\eta \\ &= (f(c_{a_{n+1}}, k_1^{m-1}))\eta = f'(c_{a_{n+1}}\eta, k_1\eta, \ldots, k_{m-1}\eta) = a_{n+1}. \end{aligned}$$

□

5. Γ-(m, n)-SEMIRING

We begin with the following definition.

Definition 5.1. Let (S, f) be a commutative m-semigroup and Γ be a non-empty set. Then, S is called a *Γ-(m, n)-semiring*, if (S, f, g) is a Γ-semigroup, that is, S satisfies the identities for all $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_m \in S$ and $x_1, x_2, \ldots, x_m \in \Gamma$,

$$g(g(a_1^{n-2}, x, a_n), y, b_3^n) = g(a_1^{n-2}, x, g(a_n, y, b_3^n))$$

$$g(a_1^{n-2}, x, f(b_1, b_2, \ldots, b_m)) = f(g(a_1^{n-2}, x, b_1), g(a_1^{n-2}, x, b_2), \ldots, g(a_1^{n-2}, x, b_m))$$

$$g(f(b_1, b_2, \ldots, b_m), x, a_3^n) = f(g(b_1, x, a_3^n), g(b_2, x, a_3^n), \ldots, g(b_m, x, a_3^n))$$

$$g(a_1^{i-1}, f(x_1, x_2, \ldots, x_m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), g(a_1^{i-1}, x_2, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_1, a_{i+1}^n)).$$

A Γ-(m, n)-semiring S is called *commutative*, if for all $a_1, a_2, \ldots, a_n \in S$, $\alpha \in \Gamma$, $i \in \{1, \ldots, n\}$ and every permutation η,

$$g(a_1^{i-1}, \alpha, a_{i+1}^n) = g(a_{\eta(1)}, a_{\eta(2)}, \ldots, a_{\eta(i-1)}\alpha, a_{\eta(i+1)}, \ldots, a_{\eta(n)}).$$

Example 5.1. We have known that $(\mathbb{N}, f)$ is a semigroup. Let $\Gamma = \{1, 2, 3\}$. For all $i \in \{1, \ldots, n\}$ define a mapping

$$h : \underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}}_{i-1} \times \Gamma \times \underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}}_{n-i} \longrightarrow \mathbb{N}$$

by $h(a_1^{i-1}, r, a_{i+1}^n) = g(a_1^{i-1}, r, a_{i+1}^n)$ for all $a_1, a_2, \ldots, a_n \in \mathbb{N}$ and $r \in \Gamma$. Then, $\mathbb{N}$ is a Γ-(m, n)-semiring.

Example 5.2. Let R be the additive commutative semiring of all $m \times n$ matrices over the set of all non-negative integers and let Γ be the additive commutative semigroup of all $n \times m$ matrices over the same set. Then, we observe that R is a Γ-$(2, 2)$-semiring.

Example 5.3. Let (S, f, g) be an arbitrary (m, n)-semiring and Γ be a non-empty set. We define a mapping

$$h : \underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}}_{i} \times \Gamma \times \underbrace{\mathbb{N} \times \mathbb{N} \times \ldots \times \mathbb{N}}_{n-i} \longrightarrow \mathbb{N}$$

by $h(a_1^i, r, a_{i+1}^n) \longrightarrow g(a_1, a_2, \ldots, a_n)$ for all $a_1, a_2, \ldots, a_n \in S$ and $r \in \Gamma$. It is easy to see that S is a Γ-(m, n)-semiring.

Thus, an (m, n)-semiring can be considered as a Γ-(m, n)-semiring.

Example 5.4. Let (S, f, g) be a Γ-(m, n)-semiring and r a fixed element in Γ. We define $h(a_1, a_2, \ldots, a_n) = g(a_1^{i-1}, r, a_{i+1}^n)$ for all $a_1, a_2, \ldots, a_n \in S$. We can show that (S, f, g) is an (m, n)-semiring.

Definition 5.1. A proper ideal P of a Γ-(m,n)-semiring (S,f,g) is said to be *prime* if for any n ideals $H_1, H_2, \ldots, H_n$ of S and $i \in \{1,\ldots,n\}$, $g(H_1^{i-1}, \Gamma, H_{i+1}^n) \subseteq P$ implies that $H_i \subseteq P$ for some i.

Let $A_1, A_2, \ldots, A_n$ be subsets of a Γ-(m,n)-semiring (S,f,g) and $\Delta \subseteq \Gamma$. We denote by $g(A_1^{i-1}, \Delta, A_{i+1}^n)$ the subset of S consisting of all finite sums of the form

$$\sum g(a_{1_j}, a_{2_j}, \ldots, a_{i-1_j}, \alpha_j, a_{i+1_j}, \ldots, a_{n_j}),$$

where $a_{1_j} \in A_1$, $a_{2_j} \in A_2$, $\ldots$, $a_{i-1_j} \in A_{i-1}$, $a_{i+1_j} \in A_{i+1}, \ldots$, $a_{n_j} \in A_n$ and $\alpha_j \in \Gamma$.

Definition 5.2. A non-empty subset T of a Γ-(m,n)-semiring (S,f,g) is called a *sub*Γ-(m,n)-*semiring* of S if T is a subsemigroup of (S,f) and $g(a_1^{i-1}, r, a_{i+1}^n) \in T$ for all $a_1, a_2, \ldots, a_n \in T$ and $r \in \Gamma$.

Definition 5.3. Let S be a Γ-(m,n)-semiring. An element $e \in S$ is called an *identity* of S if $g(e^{(i-1)}, \alpha, e^{(n-i)}) = e$ for all $\alpha \in \Gamma$.

Definition 5.4. Let X be a non-empty subset of a Γ-(m,n)-semiring S. By the term left ideal $(X)_l$ (resp. right ideal $(X)_r$, ideal $(X)_i$) of S generated by X, we mean the smallest left ideal (resp. right ideal, ideal) of S containing X, that is the intersection of all left ideals (resp. right ideals, ideals) of S containing X.

Definition 5.5. Let S be a Γ-(m,n)-semiring (S,f,g). By a quasi-ideal Q we mean a subsemigroup Q of (S,f) such that $g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, Q) \cap g(Q, S^{(i-2)}, \Gamma, S^{(n-i)}) \subseteq Q$.

It is clear that each quasi-ideal of S is a subΓ-(m,n)-semiring of S. In fact, $g(Q^{(i-1)}, \Gamma, Q^{(n-i)}) \subseteq g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, Q) \cap g(Q, S^{(i-2)}, \Gamma, S^{(n-i)}) \subseteq Q$.

Definition 5.6. Let $\mathbb{N}$ be a set of natural numbers and $\Gamma = 2\mathbb{N}$. Then, $\mathbb{N}$ is a Γ-(m,n)-semiring and $A = 3\mathbb{N}$ is a quasi-ideal of Γ-(m,n)-semiring $\mathbb{N}$.

Definition 5.7. Let X be a non-empty subset of a Γ-(m,n)-semiring S. By *quasi-ideal* $(X)_q$ *of* S *generated by* X, we mean the smallest quasi-ideal of S containing X, that is the intersection of all quasi-ideals of S containing X.

Definition 5.8. A Γ-(m,n)-semiring S is said to be a *quasi-simple* Γ-(m,n)-semiring if S is the unique quasi-ideal of S, then S has no proper quasi-ideal.

Definition 5.9. Let Q be a quasi-ideal of Γ-(m,n)-semiring (S,f,g). Then, Q is said to be *minimal quasi-ideal* of Γ-(m,n)-semiring (S,f,g) if Q does not contain any other proper quasi-ideal of S.

Theorem 5.1. *For each non-empty subset X of S the following statements hold:*

(1) $g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X)$ *is a left ideal,*
(2) $g(X, S^{(i-1)}, \Gamma, S^{(n-i-1)})$ *is a right ideal,*
(3) $g(S^{(i)}, \Gamma, S^{(j)}, X, S^{(k)}, \Gamma, S^{(n-i-j-k-3)})$ *is an ideal of S.*

Proof. (1) Suppose that

$$\begin{aligned}&g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X)\\ &= \{\sum_{j=1}^{m} g(a_{1_j}, a_{2_j}, \ldots, a_{(i-1)_j}, \alpha_j, a_{(i+1)_j}, a_{(i+2)_j}, \ldots, a_{(n-1)_j}, x_i) \mid a_{i_j} \in S,\\ &i = 1,2,3,\ldots,n, \alpha_i \in \Gamma, x_i \in X\}.\end{aligned}$$

Let $a_1, a_2, \ldots, a_m \in g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X)$. Then,

$$\begin{aligned}&f(a_1, a_2, \ldots, a_m)\\ &= \sum_{j=1}^{k} f(g(b_{1_{1j}}, b_{1_{2j}}, \ldots, b_{1_{(i-1)j}}, \alpha_{1_j}, b_{1_{(i+1)j}}, \ldots, b_{1_{(n-1)j}}, x_j),\\ &\ldots, \sum_{l=1}^{s} g(b_{m_{1j}}, b_{m_{2j}}, \ldots, b_{m_{(i-1)j}}, \alpha_{1_j}, b_{m_{(i+1)j}}, \ldots, b_{m_{(n-1)j}} x_j)),\end{aligned}$$

implies $f(a_1, a_2, \ldots, a_m)$ is a finite sum. Hence, $f(a_1, a_2, \ldots, a_m) \in g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X)$ and this shows $g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X)$ is a subsemigroup of (S,f). For $t_1, t_2, \ldots, t_n \in S$, $a \in$

$g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X)$ and $\beta \in \Gamma$, we have

$$\begin{aligned} g(t_1^{i-1}, \beta, t_{i+1}^{n-1}, a) &= g(t_1^{i-1}, \beta, t_{i+1}^{n-1}, \textstyle\sum_{j=1}^{k} g(b_{1j}, b_{2j}, \ldots, b_{(i-1)j}, \alpha_{1j}, b_{(i+1)j}, \ldots, b_{(n-1)j}, x_j)) \\ &= \sum_{j=1}^{k} g(t_1^{i-1}, \beta, t_{i+1}^{n-1}, g(b_{1j}, b_{2j}, \ldots, b_{(i-1)j}, \alpha_{1j}, b_{(i+1)j}, \ldots, b_{(n-1)j}, x_j)) \\ &= \sum_{j=1}^{k} g(g(t_1^{i-1}, \beta, t_{i+1}^{n-1}, b_{1j}), b_{2j}, \ldots, b_{(i-1)j}, \alpha_{1j}, b_{(i+1)j}, \ldots, b_{(n-1)j}, x_j)) \\ &\in g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X). \end{aligned}$$

Therefore, $g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X)$ is a left ideal of S.

(2) As in (1), we can prove that $g(X, S^{(i-1)}, \Gamma, S^{(n-i-1)})$ is a right ideal of S.

(3) By (1), $g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X)$ is a left ideal of S. Hence, we have $g(S^{(i)}, \Gamma, S^{(j)}, X, S^{(k)}, \Gamma, S^{(n-i-j-k-3)})$ is a right ideal of S by (2).

Similarly, by (2), $g(X, S^{(i-1)}, \Gamma, S^{(n-i-1)})$ is a right ideal of S. Hence, $g(S^{(i)}, \Gamma, S^{(j)}, X, S^{(k)}, \Gamma, S^{(n-i-j-k-3)})$ is a left ideal of S by (1).
Therefore, we conclude that $g(S^{(i)}, \Gamma, S^{(j)}, X, S^{(k)}, \Gamma, S^{(n-i-j-k-3)})$ is an ideal of S. □

Theorem 5.2. *Arbitrary intersection of quasi-ideals of S is either empty or a quasi-ideal of S.*

Proof. Suppose that $T = \bigcap_{i\in\Delta} \{Q_i \mid Q_i \text{ is a quasi-ideal of } S\}$, where Δ denotes any indexing set, is a non-empty set. T is a subsemigroup of (S, f). Furthermore,

$$\begin{aligned} &g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, T) \cap g(T, S^{(i-1)}, \Gamma, S^{(n-i-1)}) \\ &= g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, (\bigcap_{i\in\Delta} Q_i)) \cap g((\bigcap_{i\in\Delta} Q_i), S^{(i-1)}, \Gamma, S^{(n-i-1)}) \\ &\subseteq g(Q_i, S^{(i-1)}, \Gamma, S^{(n-i-1)}) \cap g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, Q_i) \subseteq Q_i, \end{aligned}$$

for all $i \in \Delta$. Hence, we have

$$g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, T) \cap g(T, S^{(i-1)}, \Gamma, S^{(n-i-1)}) \subseteq \bigcap_{i\in\Delta} Q_i = T.$$

This shows that T is a quasi-ideal of S. □

Theorem 5.3. *For each non-empty subset X of S, the set*

$$g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X) \cap g(X, S^{(i-1)}, \Gamma, S^{(n-i-1)})$$

is a quasi-ideal of S.

Proof. Suppose that

$$\begin{aligned} &g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X)) \\ &\cap g(g(X, S^{(i-1)}, \Gamma, S^{(n-i-1)}), S^{(i-1)}, \Gamma, S^{(n-i-1)}) \\ &= g(g(S^{(i-1)}, \Gamma, S^{(n-i)}), S^{(i-2)}, \Gamma, S^{(n-i-1)}, X) \\ &\cap g(X, S^{(i-1)}, \Gamma, S^{(n-i-2)}, g(S^{(i)}, \Gamma, S^{(n-i-1)})) \\ &\subseteq g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X) \cap g(X, S^{(i-1)}, \Gamma, S^{(n-i-1)}). \end{aligned}$$

Therefore, $g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, X) \cap g(X, S^{(i-1)}, \Gamma, S^{(n-i-1)})$ is a quasi-ideal of S. □

Theorem 5.4. *If Q is a quasi-ideal of Γ-(m, n)-semiring (S, f, g) and T is a subΓ-(m, n)-semiring of Γ-(m, n)-semiring (S, f, g), then $Q \cap T$ is a quasi-ideal of T.*

Proof. Since $Q \cap T$ is a subsemigroup of (S, f) and $Q \cap T \subseteq T$, we get $Q \cap T$ is subsemigroup of (T, f). Furthermore, we have

$$\begin{aligned} &g(T^{(i-1)}, \Gamma, T^{(n-i-1)}, (T \cap Q)) \cap g((T \cap Q), T^{(i-1)}, \Gamma, T^{(n-i-1)}) \\ &\subseteq g(T^{(i-1)}, \Gamma, T^{(n-i-1)}, Q) \cap g(Q, T^{(i-1)}, \Gamma, T^{(n-i-1)}) \\ &\subseteq g(S^{(i-1)}, \Gamma, S^{(n-i-1)}, Q) \cap g(Q, S^{(i-1)}, \Gamma, S^{(n-i-1)}) \subseteq Q, \end{aligned}$$

and

$$g(T^{(i-1)},\Gamma,T^{(n-i-1)},(T\cap Q))\cap g((T\cap Q),T^{(i-1)},\Gamma,T^{(n-i-1)})$$
$$\subseteq g(T^{(i-1)},\Gamma,T^{(n-i)})\cap g(T^{(i)},\Gamma,T^{(n-i-1)})\subseteq T\cap T=T.$$

These imply that

$$g(T^{(i-1)},\Gamma,S^{(n-i-1)},(T\cap Q))\cap g((T\cap Q),T^{(i-1)},\Gamma,S^{(n-i-1)})\subseteq Q\cap T.$$

This shows that $Q\cap T$ is a quasi-ideal of T. □

Theorem 5.5. *Intersection of a right ideal and a left ideal of Γ-(m,n)-semiring S is a quasi-ideal of S.*

Proof. Suppose that R is a right ideal and L is a left ideal of S. Then, $R\cap L$ is a subsemigroup of (S,f). Furthermore, we have

$$g(S^{(i)},\Gamma,S^{(n-i-2)},(L\cap R))\cap g((L\cap R),S^{(j)},\Gamma,S^{(n-j-2)})$$
$$=g(S^{(i)},\Gamma,S^{(n-i-2)},L)\cap g(S^{(i)},\Gamma,S^{(n-i-2)},R)\cap g(L,S^{(j)},\Gamma,S^{(n-j-2)})\cap g(R,S^{(j)},\Gamma,S^{(n-j-2)})$$
$$\subseteq g(S^{(i)},\Gamma,S^{(n-i-2)},L)\cap g(R,S^{(j)},\Gamma,S^{(n-j-2)})\subseteq L\cap R.$$

Hence, $R\cap L$ is a quasi-ideal of S. □

Theorem 5.6. *Let L be a left ideal of Γ-(m,n)-semiring S. Then, for any idempotent element e of S, $g(e,S^{(i-2)},\Gamma,S^{(n-i-1)},L)$ is a quasi-ideal of S.*

Proof. First, we prove that $g(e,S^{(i-2)},\Gamma,S^{(n-i-1)},L)=L\cap g(e,S^{(i-2)},\Gamma,S^{(n-i)})$. We know that

$$g(\underbrace{g(e,S^{(i-2)},\Gamma,S^{(n-i)}),\ldots,g(e,S^{(i-2)},\Gamma,S^{(n-i)})}_{n})\subseteq g(e,S^{(i-2)},\Gamma,S^{(n-i)}).$$

Hence, $g(e,S^{(i-2)},\Gamma,S^{(n-i)})$ is a subsemigroup of (S,f). Since

$$g(g(e,S^{(i-2)},\Gamma,S^{(n-i)}),S^{(i-1)},\Gamma,S^{(n-i-1)})$$
$$=g(e,S^{(i-2)},\Gamma,S^{(n-i-1)},g(S^{(i)},\Gamma,S^{(n-i-1)}))\subseteq g(e,S^{(i-2)},\Gamma,S^{(n-i)}),$$

$g(e,S^{(i-2)},\Gamma,S^{(n-i)})$ is a right ideal of S. Since $e\in S$ and L is a left ideal of S, it follows that $g(e,S^{(i-2)},\Gamma,S^{(n-i-1)},L)\subseteq L$. Furthermore, $g(e,S^{(i-2)},\Gamma,S^{(n-i-1)},L)\subseteq g(e,S^{(i-2)},\Gamma,S^{(n-i)})$. This implies that

$$g(e,S^{(i-2)},\Gamma,S^{(n-i-1)},L)\subseteq L\cap g(e,S^{(i-2)},\Gamma,S^{(n-i)}).$$

For the reverse inclusion let $a\in L\cap g(e,S^{(i-2)},\Gamma,S^{(n-i)})$. Hence,

$$a=\sum_{j=1}^{n}g(e,x_{2_j},x_{3j},\ldots,x_{(i-1)j},\alpha_j,x_{(i+1)j},\ldots,x_{nj}).$$

Thus, we obtain

$$a=\sum_{j=1}^{n}g(e,x_{2j},x_{3j},\ldots,x_{(i-1)j},\alpha_j,x_{(i+1)j},\ldots,x_{nj})$$
$$=\sum_{j=1}^{n}g(g(e^{(i-1)},\alpha,e^{(n-i)}),x_{2_j},x_{3_j},\ldots,x_{(i-1)j},\alpha_j,x_{(i+1)j},\ldots,x_{nj})$$
$$=g(e^{(i-1)},\alpha,e^{(n-i-1)},\sum_{j=1}^{n}g(e,x_{2j},x_{3j},\ldots,x_{(i-1)j},\alpha_j,x_{(i+1)j},\ldots,x_{nj})$$
$$=g(e^{(i-1)},\alpha,e^{(n-i-1)},a)\in g(e,S^{(i-2)},\Gamma,S^{(n-i-1)},L).$$

This shows that

$$L\cap g(e,S^{(i-2)},\Gamma,S^{(n-i)})\subseteq g(e,S^{(i-2)},\Gamma,S^{(n-i-1)},L).$$

Hence, $L\cap g(e,S^{(i-2)},\Gamma,S^{(n-i)})=g(e,S^{(i-2)},\Gamma,S^{(n-i-1)},L)$. Since L is a left ideal and

$$g(e,S^{(i-2)},\Gamma,S^{(n-i)})$$

is a right ideal of S, we conclude that $g(e,S^{(i-2)},\Gamma,S^{(n-i-1)},L)$ is a quasi-ideal of S. □

Theorem 5.7. *Let R be a right ideal of Γ-(m,n)-semiring (S,f,g). Then, for any idempotent element e of S,*

$$g(R, S^{(i-2)}, \Gamma, S^{(n-i-1)}, e)$$

is a quasi-ideal of S.

Proof. The proof is similar to the proof of Proposition 5.6. □

Theorem 5.8. *Let S be a Γ-(m,n)-semiring. Then, for any idempotent elements e, f of S,*

$$g(e, S^{(i)}, \Gamma, S^{(j-i-2)}, \Gamma, S^{(n-j-2)}, f)$$

is a quasi-ideal of S.

Proof. First, we prove that

$$g(e, S^{(i)}, \Gamma, S^{(j-i-2)}, \Gamma, S^{(n-j-2)}, f) = g(e, S^{(i)}, \Gamma, S^{(n-i-2)}) \cap g(S^{(j)}, \Gamma, S^{(n-j-2)}, f).$$

$$\begin{aligned} g(e, S^{(i)}, \Gamma, S^{(j-i-2)}, \Gamma, S^{(n-j-2)}, f) &= g(g(e, S^{(i)}, \Gamma, S^{(n-i-2)}), S^{(j-1)}, \Gamma, S^{(n-j-2)}, f) \\ &\subseteq g(e, S^{(i)}, \Gamma, S^{(n-i-2)}) \end{aligned}$$

and

$$\begin{aligned} g(e, S^{(i)}, \Gamma, S^{(j-i-2)}, \Gamma, S^{(n-j-2)}, f) &= g(e, S^{(i)}, \Gamma, S^{(n-i-3)}, g(S^{(j)}, \Gamma, S^{(n-j-2)}, f)) \\ &\subseteq g(S^{(j)}, \Gamma, S^{(n-j-2)}, f). \end{aligned}$$

Thus, we obtain

$$g(e, S^{(i)}, \Gamma, S^{(j-i-2)}, \Gamma, S^{(n-j-2)}, f) \subseteq g(e, S^{(i)}, \Gamma, S^{(n-i-2)}) \cap g(S^{(j)}, \Gamma, S^{(n-j-2)}, f).$$

Suppose that $a \in g(S^{(j)}, \Gamma, S^{(n-j-2)}, f) \cap g(e, S^{(i)}, \Gamma, S^{(n-i-2)})$. Then,

$$\begin{aligned} a &= \sum_{i=1}^{n} g(x_{1_i}, x_{2_i}, \ldots, x_{j_i}, \alpha_i, x_{(j+1)_i}, \ldots, x_{(n-2)_i}, f) \\ &= \sum_{i=1}^{n} g(x_{1_i}, x_{2_i}, \ldots, x_{j_i}, \alpha_i, x_{(j+1)_i}, \ldots, x_{(n-2)_i}, g(f^{(k)}, \alpha, f^{(n-k-1)})) \\ &= \sum_{i=1}^{n} g(g(x_{1_i}, x_{2_i}, \ldots, x_{j_i}, \alpha_i, x_{(j+1)_i}, \ldots, x_{(n-2)_i}, f), f^{(k-1)}, \alpha, f^{(n-k-1)}) \\ &= g(a, f^{(k-1)}, \alpha, f^{(n-k-1)}). \end{aligned}$$

Hence, $a = g(a, f^{(k-1)}, \alpha, f^{(n-k-1)})$ for all $\alpha \in \Gamma$. Since $a \in g(e, S^{(i-2)}, \Gamma, S^{(n-i)})$, $\alpha \in \Gamma$, it follows that

$$a = g(a, f^{(k-1)}, \alpha, f^{(n-k-1)}) \in g(e, S^{(i)}, \Gamma, S^{(j-i-2)}, \Gamma, S^{(n-j-2)}, f).$$

We obtain

$$g(e, S^{(i)}, \Gamma, S^{(n-i-2)}) \cap g(S^{(j)}, \Gamma, S^{(n-j-2)}, f) \subseteq g(e, S^{(i)}, \Gamma, S^{(j-i-2)}, \Gamma, S^{(n-j-2)}, f).$$

Thus, we have

$$g(e, S^{(i)}, \Gamma, S^{(n-i-2)}) \cap g(S^{(j)}, \Gamma, S^{(n-j-2)}, f) = g(e, S^{(i)}, \Gamma, S^{(j-i-2)}, \Gamma, S^{(n-j-2)}, f).$$

Since $g(S^{(j)}, \Gamma, S^{(n-j-2)}, f)$ is a left ideal and $g(e, S^{(i)}, \Gamma, S^{(n-i-2)})$ is a right ideal of S, we get

$$g(e, S^{(i)}, \Gamma, S^{(n-i-2)}) \cap g(S^{(j)}, \Gamma, S^{(n-j-2)}, f) = g(e, S^{(i)}, \Gamma, S^{(j-i-2)}, \Gamma, S^{(n-j-2)}, f)$$

is a quasi-ideal of S. □

Theorem 5.9. *If (S,f,g) is a Γ-(m,n)-semiring, then S is a quasi-simple Γ-(m,n)-semiring if and only if $g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \cap g(a, S^{(j)}, \Gamma, S^{(n-j-2)}) = S$ for all $a \in S$.*

Proof. Suppose that S is a quasi-simple Γ-(m,n)-semiring. For every $a \in S$, $g(S^{(i)}, \Gamma, S^{(n-i-2)}, a)$ and $g(a, S^{(j)}, \Gamma, S^{(n-j-2)})$ are left and right ideals of S, respectively. Therefore,

$$g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \cap g(a, S^{(j)}, \Gamma, S^{(n-j-2)})$$

is a quasi-ideal of S. Furthermore, $g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \subseteq S$ and $g(a, S^{(j)}, \Gamma, S^{(n-j-2)}) \subseteq S$ imply $g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \cap g(a, S^{(j)}, \Gamma, S^{(n-j-2)}) \subseteq S$. Since S is a quasi-simple Γ-(m,n)-semiring, it follows that $S = g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \cap g(a, S^{(i)}, \Gamma, S^{(n-i-2)})$.

Conversely, suppose that $S = g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \cap g(a, S^{(j)}, \Gamma, S^{(n-j-2)})$. Let Q be a quasi-ideal of S. For any $q \in Q$, by assumption we have,

$$S = g(S^{(i)}, \Gamma, S^{(n-i-2)}, q) \cap g(q, S^{(j)}, \Gamma, S^{(n-j-2)}) \subseteq$$
$$g(S^{(i)}, \Gamma, S^{(n-j-2)}, Q) \cap g(Q, S^{(j)}, \Gamma, S^{(n-j-2)}) \subseteq Q.$$

Therefore, $S \subseteq Q$. Thus $S = Q$. Hence, S is a quasi-simple Γ-(m,n)-semiring. □

Theorem 5.10. *The intersection of a minimal right ideal and a minimal left ideal of a* Γ-(m,n)-*semiring* S *is a minimal quasi-ideal of* S.

Proof. Let R and L denote the minimal right ideal and the minimal left ideal of S, respectively. Define $Q = R \cap L$. Then, Q is a quasi-ideal of S. Let Q_1 be a quasi-ideal of S such that $Q_1 \subseteq Q$. Then, $g(S^{(i)}\Gamma, S^{(n-i-2)}, Q_1)$ is a left ideal and $g(Q_1, S^{(i)}, \Gamma, S^{(n-i-2)})$ is a right ideal of S. So, $Q_1 \subseteq L$ implies

$$g(S^{(i)}, \Gamma, S^{(n-i-2)}, Q_1) \subseteq g(S^{(i)}, \Gamma, S^{(n-i-2)}, L) \subseteq L.$$

Also, $Q_1 \subseteq R$ implies

$$g(Q_1, S^{(j)}, \Gamma, S^{(n-j-2)}) \subseteq g(R, S^{(j)}, \Gamma, S^{(n-j-2)}) \subseteq R.$$

By the minimality of R and L, we have

$$g(S^{(i)}, \Gamma, S^{(n-i-2)}, Q_1) = L$$

and

$$g(Q_1, S^{(j)}, \Gamma, S^{(n-j-2)}) = R.$$

Therefore, we have

$$Q = R \cap L = g(S^{(i)}, \Gamma, S^{(n-i-2)}, Q_1) \cap g(Q_1, S^{(j)}, \Gamma, S^{(n-j-2)}) \subseteq Q_1.$$

Hence, $Q_1 = Q$. This shows that Q is a minimal quasi-ideal of S. □

Theorem 5.11. *If* Q *is a minimal quasi-ideal of* Γ-(m,n)-*semiring* S, *then any two non-zero elements of* Q *generate the same left (right) ideal of* S.

Proof. Let Q be a minimal quasi-ideal of S and x be a non-zero element of Q. Then, $(x)_l$, the left ideal generated by x, is a quasi-ideal of S. Hence, $(x)_l \cap Q$ is a quasi-ideal of S. As $(x)_l \cap Q \subseteq Q$ and Q is a minimal quasi-ideal of S we get $(x)_l \cap Q = Q$. Thus, $Q \subseteq (x)_l$. For any non-zero element y of Q, $y \in Q$ implies $y \in (x)_l$. Therefore, $(y)_l \subseteq (x)_l$. Similarly, we can show that $(x)_l \subseteq (y)_l$. Hence, $(x)_l = (y)_l$.

In the same way, we can prove that any two non-zero elements of Q generate the same right ideal of S. □

Theorem 5.12. *Let* Q *be a quasi-ideal of* Γ-(m,n)-*semiring* S. *If* Q *itself is a quasi-simple* Γ-(m,n)-*semiring, then* Q *is a minimal quasi-ideal of* S.

Proof. Since Q is a quasi-ideal of S, it follows that Q is a subΓ-(m,n)-semiring of S. Suppose that Q is a quasi-simple Γ-(m,n)-semiring. Let Q_1 be a quasi-ideal of S such that $Q_1 \subseteq Q$. Then, we obtain

$$g(Q^{(i)}, \Gamma, Q^{(n-i-2)}, Q_1) \cap g(Q_1, Q^{(i)}, \Gamma, Q^{(n-i-2)}) \subseteq$$
$$g(S^{(i)}, \Gamma, S^{(n-i-2)}, Q_1) \cap g(Q_1, S^{(i)}, \Gamma, S^{(n-i-2)}) \subseteq Q_1.$$

Therefore, Q_1 is a quasi-ideal of Q. Since $Q_1 \subseteq Q$, Q_1 is a quasi-ideal of Q and Q is a quasi-simple Γ-(m,n)-semiring, it follows that $Q_1 = Q$. Therefore, Q is a minimal quasi-ideal of S. □

Theorem 5.13. *Every minimal quasi-ideal Q of Γ-(m,n)-semiring (S,f,g) is represented as*

$$Q = g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \cap g(a, S^{(i)}, \Gamma, S^{(n-i-2)}),$$

where a is any element of Q, $g(S^{(i)}, \Gamma, S^{(n-i-2)}, a)$ and $g(a, S^{(i)}, \Gamma, S^{(n-i-2)})$ is a minimal left ideal and a minimal right ideal of S, respectively.

Proof. Suppose that Q is a minimal quasi-ideal of S and $a \in Q$. Then, $g(S^{(i)}, \Gamma, S^{(n-i-2)}, a)$ and $g(a, S^{(i)}, \Gamma, S^{(n-i-2)})$ is a left ideal and a right ideal of S, respectively. Therefore, we conclude that $g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \cap g(a, S^{(i)}, \Gamma, S^{(n-i-2)})$ is a quasi-ideal of S. Then

$$\begin{aligned} g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \cap g(a, S^{(i)}, \Gamma, S^{(n-i-2)}) &\subseteq g(S^{(i)}, \Gamma, S^{(n-i-2)}, Q) \cap g(Q, S^{(i)}, \Gamma, S^{(n-i-2)}) \\ &\subseteq Q. \end{aligned}$$

By the minimality of Q, we obtain $Q = g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \cap g(a, S^{(i)}, \Gamma, S^{(n-i-2)})$. Now, in order to show that $g(S^{(i)}, \Gamma, S^{(n-i-2)}, a)$ is a minimal left ideal, let L be a left ideal of S such that $L \subseteq g(S^{(i)}, \Gamma, S^{(n-i-2)}, a)$. Then,

$$g(S^{(i)}, \Gamma, S^{(n-i-2)}, L) \subseteq L \subseteq g(S^{(i)}, \Gamma, S^{(n-i-2)}, a),$$

$$g(S^{(i)}, \Gamma, S^{(n-i-2)}, L) \cap g(a, S^{(i)}, \Gamma, S^{(n-i-2)}) \subseteq g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \cap g(a, S^{(i)}, \Gamma, S^{(n-i-2)}) = Q.$$

Since $g(S^{(i)}, \Gamma, S^{(n-i-2)}, L)$ is a left ideal of S and $g(a, S^{(i)}, \Gamma, S^{(n-i-2)})$ is a right ideal of S, we conclude that $g(S^{(i)}, \Gamma, S^{(n-i-2)}, L) \cap g(a, S^{(i)}, \Gamma, S^{(n-i-2)})$ is a quasi-ideal of S. Furthermore, since $g(S^{(i)}, \Gamma, S^{(n-i-2)}, L) \cap g(a, S^{(i)}, \Gamma, S^{(n-i-2)}) \subseteq Q$ and Q is minimal quasi-ideal of S, we have $Q = g(S^{(i)}, \Gamma, S^{(n-i-2)}, L) \cap g(a, S^{(i)}, \Gamma, S^{(n-i-2)}) \subseteq g(S^{(i)}, \Gamma, S^{(n-i-2)}, L)$. Now, we have

$$\begin{aligned} g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) &\subseteq g(S^{(i)}, \Gamma, S^{(n-i-2)}, Q) \subseteq g(S^{(i)}, \Gamma, S^{(n-i-2)}, g(S^{(i)}, \Gamma, S^{(n-i-2)}, L)) \\ &= g(g(S^{(i)}, \Gamma, S^{(n-i-1)}), S^{(i-1)}, \Gamma, S^{(n-i-2)}, L) \\ &\subseteq g(S^{(i)}, \Gamma, S^{(n-i-2)}, L) \subseteq L. \end{aligned}$$

This shows that $g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) \subseteq L$. Therefore, $g(S^{(i)}, \Gamma, S^{(n-i-2)}, a) = L$. Hence, $g(S^{(i)}, \Gamma, S^{(n-i-2)}, a)$ is a minimal left ideal of S. Similarly, we can prove that $g(a, S^{(i)}, \Gamma, S^{(n-i-2)})$ is a minimal right ideal of S. □

6. Conclusions

Semirings constitute a natural generalization of rings with broad applications in the mathematical foundation of computer sciences. The class of (m,n)-semirings is a generalization of semirings. We studied special ideals hand homomorphisms of (m,n)-semirings. In particular, we studied Γ-(m,n)-semirings and investigated their properties.

For future research, one may consider (m,n)-semihyperrings and related algebraic structures and study their properties.

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